The Cauchy Integral

Analyticity of $\oint_\gamma \varphi(\xi)/(\xi - z)\, d\xi$

Lecture Notes

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Reminder (power series)

Let $f(w) := \sum_{n=0}^{\infty} a_n w^n$, with radius of convergence R. Then f is analytic for $|w| < R$ and may be differentiated term by term:

$$f^{(k)}(w) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n w^{n-k}, \qquad a_n = \frac{f^{(n)}(0)}{n!}.$$

Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth curve, and let φ be bounded and integrable along γ (for example, continuous or piecewise continuous along the curve). We also write γ for its image; the curve need not be closed. For $z \notin \gamma$, define

$$F(z) := \oint_\gamma \frac{\varphi(\xi)}{\xi - z}\, d\xi.$$

Theorem (analyticity of the Cauchy integral)

At every point of $\mathbb{C} \setminus \gamma$, F is locally the sum of a convergent power series; hence F is analytic and **infinitely differentiable**, with complex derivatives of all orders. If $z_0 \notin \gamma$ and $R := \operatorname{dist}(z_0, \gamma) > 0$, then

$$F(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \qquad (|z - z_0| < R),$$

where, for every integer $k \geq 0$,

$$a_k = \frac{F^{(k)}(z_0)}{k!} = \oint_\gamma \frac{\varphi(\xi)}{(\xi - z_0)^{k+1}}\, d\xi.$$

1 The Cauchy Trick: A Finite Expansion

Fix $z_0 \notin \gamma$, $z \notin \gamma$, and $\xi \in \gamma$. Since $\xi - z_0 \neq 0$ and $\xi - z \neq 0$, the number $\lambda := (z - z_0)/(\xi - z_0)$ is defined and is not equal to 1. For every integer $n \geq 1$, the finite geometric-series identity gives

$$\frac{1}{1 - \lambda} = \sum_{k=0}^{n-1} \lambda^k + \frac{\lambda^n}{1 - \lambda}.$$

Substituting the value of λ, we obtain

$$\frac{1}{1 - \dfrac{z - z_0}{\xi - z_0}} = \sum_{k=0}^{n-1} \left(\frac{z - z_0}{\xi - z_0}\right)^k + \frac{\left(\dfrac{z - z_0}{\xi - z_0}\right)^n}{1 - \dfrac{z - z_0}{\xi - z_0}}.$$

Since $1-(z-z_0)/(\xi-z_0)=(\xi-z)/(\xi-z_0)$, this is equivalent to

$$\frac{\xi-z_0}{\xi-z}=\sum_{k=0}^{n-1}\left(\frac{z-z_0}{\xi-z_0}\right)^k+\frac{(z-z_0)^n}{(\xi-z_0)^{n-1}(\xi-z)}.$$

Dividing by $\xi-z_0$ gives

$$\boxed{\frac{1}{\xi-z}=\sum_{k=0}^{n-1}\frac{(z-z_0)^k}{(\xi-z_0)^{k+1}}+\frac{(z-z_0)^n}{(\xi-z_0)^n(\xi-z)}.}$$

Multiplying by $\varphi(\xi)$ and integrating the finite sum over γ, we obtain

$$\begin{aligned}F(z)&=\oint_\gamma\frac{\varphi(\xi)}{\xi-z}\,d\xi\\&=\sum_{k=0}^{n-1}\oint_\gamma(z-z_0)^k\frac{\varphi(\xi)}{(\xi-z_0)^{k+1}}\,d\xi+\oint_\gamma(z-z_0)^n\frac{\varphi(\xi)}{(\xi-z_0)^n(\xi-z)}\,d\xi\\&=\sum_{k=0}^{n-1}a_k(z-z_0)^k+(z-z_0)^nF_n(z),\end{aligned}$$

where

$$a_k:=\oint_\gamma\frac{\varphi(\xi)}{(\xi-z_0)^{k+1}}\,d\xi,\qquad F_n(z):=\oint_\gamma\frac{\varphi(\xi)}{(\xi-z_0)^n(\xi-z)}\,d\xi.$$

Here $z-z_0$ is constant with respect to the integration variable ξ, so its powers can be taken outside the integrals. This finite identity holds for *every* $z\notin\gamma$; it does not require $|z-z_0|<R$.

2 Convergence of the Remainder

Now suppose $s:=|z-z_0|<R:=\operatorname{dist}(z_0,\gamma)$. To pass from the finite identity to the infinite series, it suffices to prove

$$\lim_{n\to\infty}(z-z_0)^nF_n(z)=\lim_{n\to\infty}\oint_\gamma\frac{(z-z_0)^n\varphi(\xi)}{(\xi-z_0)^n(\xi-z)}\,d\xi=0.$$

Put $q:=s/R<1$, let $M:=\sup_{\xi\in\gamma}|\varphi(\xi)|<\infty$, and let $L:=\ell(\gamma)$. We use the following four estimates, valid for every $\xi\in\gamma$:

1. The ratio in the geometric remainder satisfies

$$\frac{|z-z_0|}{|\xi-z_0|}\le\frac{s}{R}=q<1.$$

2. By the definition of the distance from z_0 to γ,

$$|\xi-z_0|\ge R>0.$$

3. By the reverse triangle inequality,

$$|\xi-z|=|(\xi-z_0)-(z-z_0)|\ge|\xi-z_0|-|z-z_0|\ge R-s>0.$$

4. Boundedness of the density gives

$$|\varphi(\xi)|\le M.$$

Consequently, the integrand of the remainder has the pointwise bound

$$\left|\frac{(z-z_0)^n\varphi(\xi)}{(\xi-z_0)^n(\xi-z)}\right| = \left|\frac{z-z_0}{\xi-z_0}\right|^n \frac{|\varphi(\xi)|}{|\xi-z|} \le q^n \frac{M}{R-s}.$$

Applying the integral inequality and then integrating this constant bound with respect to arc length gives

$$\begin{aligned}
|z-z_0|^n|F_n(z)| &= \left|\oint_\gamma \frac{(z-z_0)^n\varphi(\xi)}{(\xi-z_0)^n(\xi-z)}\,d\xi\right| \\
&\le \int_\gamma \frac{|z-z_0|^n|\varphi(\xi)|}{|\xi-z_0|^n|\xi-z|}\,|d\xi| \\
&\le \int_\gamma q^n \frac{M}{R-s}\,|d\xi| \\
&= q^n \frac{M}{R-s}\,L = \frac{ML}{R-s}\left(\frac{s}{R}\right)^n \longrightarrow 0.
\end{aligned}$$

The convergence follows because $0 \le q < 1$, while M, L, and $R-s>0$ are fixed as $n \to \infty$. Passing to the limit in the finite expansion therefore proves

$$F(z) = \lim_{n\to\infty} \sum_{k=0}^{n-1} a_k(z-z_0)^k = \sum_{k=0}^{\infty} a_k(z-z_0)^k \qquad (|z-z_0| < R).$$

3 Taylor Coefficients and Derivatives

By the power-series theorem, F is analytic and has complex derivatives of every order in $B(z_0,R)$. Since z_0 was arbitrary off γ, this proves analyticity and infinite differentiability on $\mathbb{C}\setminus\gamma$. Termwise differentiation of the obtained power series at its centre z_0 gives, for every integer $k \ge 0$,

$$\boxed{F^{(k)}(z_0) = k!a_k = k!\oint_\gamma \frac{\varphi(\xi)}{(\xi-z_0)^{k+1}}\,d\xi.}$$

Thus the coefficients are the Taylor coefficients of F. The finite expansion expresses F as its Taylor polynomial of degree at most $n-1$ plus the remainder $(z-z_0)^nF_n(z)$ in **Cauchy form**. ■